

The Aharonov–Bohm effect in a spatially confining theory based on a turbulent fluid

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Wilson loops in a turbulent fluid are shown to respect a specific area law corresponding to the Kolmogorov scaling. This law leads to the condensation of a complex-valued scalar field minimally coupled to the velocity field. We use this finding to estimate a v.e.v. of the dual Higgs field, which appears in the hydrodynamic description of a spatially confining dual Landau–Ginzburg theory. The temperature dependence of all other parameters of this theory is found upon a comparison with the spatial string tension and the chromo-magnetic vacuum correlation length of the Yang–Mills gluon plasma. In particular, a nonperturbative contribution to the shear viscosity of the dual fluid comes out exponentially suppressed with temperature. Interactions of the dual Abrikosov vortices with excitations of the fluid yield a long-range Aharonov–Bohm effect. This effect is shown to take place for all but calculated discrete values of the product of the kinematic viscosity of the fluid to the coupling constant of the dual Higgs and velocity fields. In the presence of a Chern–Simons term with the parameter Θ , these discrete values remain the same for sufficiently large values of Θ . Furthermore, the Chern–Simons term leads to the appearance of knotted dual Abrikosov vortices.

I. INTRODUCTION

The classic 't Hooft–Mandelstam scenario of confinement is based on the condensation of magnetic monopoles [1]. This phenomenon is described in terms of a magnetically-charged Higgs field. In case of the simplest, $SU(2)$, Yang–Mills theory, the resulting Abelian Higgs model is just a 4-d generalization of the Landau–Ginzburg type-II superconductor, whose gauge field is dual to the A_μ^3 -gluon of the initial Yang–Mills theory. It is also generally known that, in the deconfinement phase of the Yang–Mills theory, large spatial Wilson loops keep

exhibiting an area-law behavior, which is the essence of spatial confinement (see Ref. [2] for the corresponding lattice results in the SU(2) Yang–Mills theory). Analytically, spatial confinement can with a good accuracy be described in terms of soft stochastic chromo-magnetic Yang–Mills fields [3]. Thus, above a certain temperature of the dimensional reduction, the soft part of the SU(2) Yang–Mills gluon plasma can be modeled by means of a dual Landau–Ginzburg theory with temperature-dependent couplings.

On the other hand, it is a well-known property of a conducting turbulent medium that it expels an external magnetic field [4]. Superimposing this fact with the ’t Hooft–Mandelstam scenario of spatial confinement, one can guess that the dual-Higgs condensate, which within this scenario expels external chromo-electric fields, is formed owing to some dual turbulent medium. Such a medium implies an effective hydrodynamic description of the said dual Landau–Ginzburg theory, i.e. a description where the dual gauge field is substituted by a velocity field v_μ . Recently [5], the ratio of the shear viscosity to the entropy density in a medium of magnetically-charged hydrodynamic fluctuations has been found to be much smaller than unity. We note also that a formally similar model, where the velocity plays a role of the gauge field interacting with quarks, was studied recently in Ref. [6].

In this paper, we consider condensation of the dual Higgs field interacting with the dual velocity v_μ similarly to the formation of a heavy-quark condensate in QCD, which occurs due to the interaction of this quark with soft stochastic Yang–Mills fields [7]. To this end, we first find an area-dependence of the Wilson loop corresponding to the Kolmogorov scaling law in turbulence [8]. We consider then a complex-valued scalar field, which interacts via a covariant derivative with the velocity field. Using in the effective action of this complex-valued field the so-obtained Wilson loop, we get an estimate for the v.e.v. of the dual Higgs field. Using furthermore the known temperature behaviors of the spatial string tension and the chromo-magnetic vacuum correlation length in the Yang–Mills theory, we evaluate also the kinematic viscosity of the dual fluid and the phenomenological coupling of the dual Higgs and velocity fields. This procedure fixes completely the temperature dependence of all the parameters entering the effective hydrodynamic description the dual Landau–Ginzburg theory. Having these parameters fixed, we proceed to the analysis of topological effects stemming from the interaction of excitations of the dual fluid with the dual (i.e. carrying electric fluxes) Abrikosov vortices [9], which are present in the dual Landau–Ginzburg theory. To this end, we perform a duality transformation of the Wilson loop describing such an excitation, and

find in particular a long-range Aharonov–Bohm-type interaction between excitations of the dual fluid and the dual Abrikosov vortices. Finally, we extend this analysis to the case where a Chern–Simons (CS) term is present.

The paper is organized as follows. In the next Section, we derive for the Wilson loop in hydrodynamics a novel area law corresponding to the Kolmogorov scaling. In Section III, we consider an interaction of the velocity field, which respects this scaling, with a complex-valued scalar field. Using the obtained specific area law for the Wilson loop, we evaluate a v.e.v. of the dual Higgs field in terms of the energy dissipation rate and the kinematic viscosity of the dual fluid. We make further a natural assumption that the dual Landau–Ginzburg theory represents a dimensionally-reduced high-temperature phase of a 4-d dual Abelian Higgs model. In the London limit, this model yields a string tension and a vacuum correlation length, which can be compared with the known spatial string tension and the chromo-magnetic vacuum correlation length of the Yang–Mills gluon plasma. Using this correspondence, we obtain the kinematic viscosity of the dual fluid in terms of the following three quantities: the energy dissipation rate, the temperature, and the finite-temperature Yang–Mills coupling. In the same way, we also obtain in terms of these three quantities a dimensionful phenomenological coupling μ of the dual Higgs and velocity fields. In particular, the obtained purely nonperturbative kinematic viscosity turns out to be exponentially suppressed with temperature.

As was mentioned above, the Landau–Ginzburg theory at issue possesses dual Abrikosov vortices, which interact with excitations of the dual fluid. In Section IV, we show that among these interactions there exists a long-range Aharonov–Bohm one, and find a relation between the kinematic viscosity and the coupling μ , for which this interaction disappears. In Section V, we additionally consider the effects produced by the CS term. First, we briefly show that, in the absence of the dual Higgs field, the CS term yields a self-linking of the contour along which the dual excitation evolves. Then we consider an excitation of the dual fluid in the full theory, which includes also the dual Higgs field, and obtain there the Aharonov–Bohm-type interactions in the presence of a CS term. In particular, at sufficiently large values of the Θ -parameter entering the CS term, we obtain an analytic expression for the Wilson loop associated with an excitation of the dual fluid. Using this expression, we find a relation between the kinematic viscosity and the coupling μ , for which the Aharonov–Bohm effect in the large- Θ limit disappears. Remarkably, this relation turns

out to be the same as the one obtained in Section IV in the absence of the CS term. We also notice that the temperature behaviors of the kinematic viscosity and the coupling μ , obtained in Section III, allow this relation to hold, that makes it nontrivial. Furthermore, we explicitly find the appearance of knotted dual Abrikosov vortices in the same large- Θ limit. In Section VI, the summary of the results obtained is presented. In Appendices A and B, we provide some technical details of the calculations performed.

II. A NOVEL AREA LAW OF THE WILSON LOOP IN TURBULENCE FROM THE KOLMOGOROV SCALING

We start with considering a Wilson loop in turbulence [10],

$$\langle W(C) \rangle = \left\langle \exp \left(\frac{i}{\nu} \int d^3x v_\mu j_\mu \right) \right\rangle, \quad (1)$$

and derive its area dependence corresponding to the Kolmogorov scaling law. In Eq. (1), $j_\mu(\mathbf{x}; C) = \oint_C dx_\mu(\tau) \delta(\mathbf{x} - \mathbf{x}(\tau))$ is a conserved current, and ν is the kinematic viscosity of the fluid, equal to the ratio of the shear viscosity to the density. The Kolmogorov hypothesis [8] states that the mean dissipation of the kinetic energy by a unit mass of the fluid, occurring in the unit time, is constant: $\mathcal{E} = \text{const}$. Since $\mathcal{E} = v^2/t$, where $t \sim |\mathbf{x}|/v$, one gets the Kolmogorov scaling law:

$$v \sim (\mathcal{E}|\mathbf{x}|)^{1/3}, \quad (2)$$

where $v = |\mathbf{v}|$. This law can be given a field-theoretical description by evaluating v as $\sqrt{\langle v_\mu^2 \rangle_{\mathcal{F}}}$. Here $\langle \cdots \rangle_{\mathcal{F}}$ effectively substitutes the average $\langle \cdots \rangle$ from Eq. (1) at sufficiently small distances, where the Kolmogorov scaling holds. At such distances, one can neglect connected velocities' correlation functions higher than the two-point one, and approximate $\langle W(C) \rangle_{\mathcal{F}}$ as

$$\langle W(C) \rangle_{\mathcal{F}} \simeq \exp \left(-\frac{1}{2\nu^2} \oint_C dx_\mu \oint_C dy_\nu \langle v_\mu(\mathbf{x}) v_\nu(\mathbf{y}) \rangle_{\mathcal{F}} \right) = \exp \left(-\text{const} \cdot \frac{\mathcal{E}^{2/3}}{\nu^2} S^{4/3} \right), \quad (3)$$

where S is the area of the minimal surface encircled by the non-selfintersecting contour C .

The average $\langle \cdots \rangle_{\mathcal{F}}$ can be formally defined by representing v_μ in the form $v_\mu = \frac{x_\nu}{|\mathbf{x}|^{2/3}} \mathcal{F}_{\nu\mu}$, where $\mathcal{F}_{\nu\mu}$ is a space-independent matrix with the Gaussian distribution:

$$\langle \cdots \rangle_{\mathcal{F}} = \frac{1}{(\pi\gamma)^{n/2}} \left(\prod_{\mu < \nu} \int_{-\infty}^{+\infty} d\mathcal{F}_{\mu\nu} e^{-\frac{\mathcal{F}_{\mu\nu}^2}{\gamma}} \right) (\cdots). \quad (4)$$

Here, in the general case of a d -dimensional Euclidean space, $n = (d^2 - d)/2$, and the average is normalized by the standard condition $\langle 1 \rangle_{\mathcal{F}} = 1$. Noticing that

$$\langle \mathcal{F}_{\mu\nu}^2 \rangle_{\mathcal{F}} = \frac{2}{(\pi\gamma)^{n/2}} \left(\prod_{\mu < \nu} \int_{-\infty}^{+\infty} d\mathcal{F}_{\mu\nu} e^{-\frac{\mathcal{F}_{\mu\nu}^2}{\gamma}} \mathcal{F}_{\mu\nu}^2 \right) = -\frac{2}{(\pi\gamma)^{n/2}} \frac{\partial}{\partial \gamma^{-1}} (\pi\gamma)^{n/2} = n\gamma,$$

we can obtain the coefficient c in the formula $\langle \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\rho} \rangle_{\mathcal{F}} = c \cdot (\delta_{\mu\lambda} \delta_{\nu\rho} - \delta_{\mu\rho} \delta_{\nu\lambda})$. Indeed, this formula yields $\langle \mathcal{F}_{\mu\nu}^2 \rangle_{\mathcal{F}} = c \cdot (d^2 - d)$, that leads to the value of $c = \frac{n\gamma}{d^2 - d} = \frac{\gamma}{2}$. Therefore, setting $d = 3$, one has

$$\langle v_{\mu}^2 \rangle_{\mathcal{F}} = \frac{x_{\nu} x_{\lambda}}{|\mathbf{x}|^{4/3}} \langle \mathcal{F}_{\nu\mu} \mathcal{F}_{\lambda\mu} \rangle_{\mathcal{F}} = \gamma |\mathbf{x}|^{2/3},$$

where the above-obtained mean value $\langle \mathcal{F}_{\mu\nu} \mathcal{F}_{\lambda\rho} \rangle_{\mathcal{F}} = \frac{\gamma}{2} (\delta_{\mu\lambda} \delta_{\nu\rho} - \delta_{\mu\rho} \delta_{\nu\lambda})$ has been used at the final step. Equating now $\sqrt{\langle v_{\mu}^2 \rangle_{\mathcal{F}}} = \gamma^{1/2} |\mathbf{x}|^{1/3}$ to the right-hand side of Eq. (2), we obtain from the Kolmogorov scaling law the coefficient γ in terms of the energy dissipation rate:

$$\gamma \sim \mathcal{E}^{2/3}.$$

Furthermore, using the correlation function

$$\langle v_{\mu}(\mathbf{x}) v_{\nu}(\mathbf{y}) \rangle_{\mathcal{F}} = \frac{\gamma}{2} \frac{x_{\lambda} y_{\rho}}{|\mathbf{x}|^{2/3} |\mathbf{y}|^{2/3}} (\delta_{\lambda\rho} \delta_{\mu\nu} - \delta_{\lambda\nu} \delta_{\mu\rho}), \quad (5)$$

one gets for the Wilson loop (1) the following expression:

$$\begin{aligned} \langle W(C) \rangle_{\mathcal{F}} &= \exp \left(-\frac{1}{2\nu^2} \oint_C dx_{\mu} \oint_C dy_{\nu} \langle v_{\mu}(\mathbf{x}) v_{\nu}(\mathbf{y}) \rangle_{\mathcal{F}} \right) = \\ &= \exp \left\{ -\frac{\gamma}{4\nu^2} \left[\oint_C dx_{\mu} \oint_C dy_{\mu} \frac{\mathbf{x}\mathbf{y}}{|\mathbf{x}|^{2/3} |\mathbf{y}|^{2/3}} - \oint_C \frac{dx_{\mu} x_{\nu}}{|\mathbf{x}|^{2/3}} \oint_C \frac{dy_{\nu} y_{\mu}}{|\mathbf{y}|^{2/3}} \right] \right\}. \end{aligned}$$

For a flat contour, the expression in the square brackets can be estimated as $[\dots] \sim S^{4/3}$. (In particular, an elementary calculation yields for a circle $[\dots] = 4\pi^{2/3} S^{4/3}$.) Thus, expression (3) for the Wilson loop is recovered.

III. A HYDRODYNAMIC DESCRIPTION OF A SPATIALLY CONFINING DUAL LANDAU–GINZBURG THEORY

A. Condensation of a complex scalar field interacting with the turbulent fluid

Let us consider a complex-valued scalar field ϕ of some mass M , interacting with the turbulent fluid. The action of such a field has the form $S = \int d^3x (|D_{\mu}\phi|^2 + M^2|\phi|^2)$, where

the covariant derivative reads $D_\mu = \partial_\mu - \frac{i}{\nu} v_\mu$. In the course of its spatial evolution, the field ϕ yields the Wilson loop (1). We show how the specific area law in turbulence, Eq. (3), leads to the condensation of the field ϕ , i.e. to the formation of the condensate $\langle |\phi|^2 \rangle$. (From now on, we denote by $\langle \dots \rangle$ the average $\langle \dots \rangle_{\mathcal{F}}$.)

To this end, we integrate in the partition function over the ϕ - and the ϕ^* -fields, that yields the one-loop effective action

$$\langle \Gamma[v_\mu] \rangle = \left\langle \text{tr} \ln \frac{(-D_\mu^2 + M^2)}{(-\partial_\mu^2 + M^2)} \right\rangle.$$

We use here the normalization condition $\langle \Gamma[0] \rangle = 0$, which corresponds to our assumption that the ϕ -field is condensed because of its interaction with the velocities v_μ 's respecting the Kolmogorov scaling. One can further represent $\langle \Gamma[v_\mu] \rangle$ as the following path integral:

$$\langle \Gamma[v_\mu] \rangle = 2V \int_0^\infty \frac{ds}{s} e^{-M^2 s} \left[\int_{x_\mu(0)=x_\mu(s)=0} \mathcal{D}x_\mu e^{-\frac{1}{4} \int_0^s d\tau \dot{x}_\mu^2} \langle W(C) \rangle - \frac{1}{(4\pi s)^{3/2}} \right]. \quad (6)$$

Here $\langle W(C) \rangle$ is given by Eq. (3), the factor of 2 stems from the complex-valuedness of ϕ , and V is a 3-d volume occupied by the system. The condensate of the ϕ -field can then be found from the relation

$$\langle |\phi|^2 \rangle = -\frac{1}{V} \frac{\partial}{\partial M^2} \langle \Gamma[v_\mu] \rangle. \quad (7)$$

We calculate the path integral that enters Eq. (6) by reducing it to the one in an auxiliary space-independent electromagnetic field of a geometric origin. To this end, we use the fact that, for a sufficiently small non-selfintersecting contour C at issue, the area S of the minimal surface encircled by such a contour can be represented in the form $S = \sqrt{\Sigma_{\mu\nu}^2}/2$, where $\Sigma_{\mu\nu} = \oint_C dx_\mu x_\nu$ is the so-called tensor area [10]. Furthermore, we use an elementary saddle-point integral

$$e^{-\kappa^{2/3}} \simeq \sqrt{\frac{3}{\pi}} \int_0^\infty d\lambda e^{-\lambda^2 - \frac{2\kappa}{3^{3/2}\lambda}}, \quad \text{where } \kappa > 0,$$

owing to which we can represent the Wilson loop (3) in the form

$$\langle W(C) \rangle \simeq \sqrt{\frac{3}{\pi}} \int_0^\infty d\lambda e^{-\lambda^2 - G(\lambda) \cdot \Sigma_{\mu\nu}^2}. \quad (8)$$

In this formula, we have denoted

$$G(\lambda) \equiv \left(\frac{\text{const}}{3} \right)^{3/2} \cdot \frac{\mathcal{E}}{\nu^3 \lambda}, \quad (9)$$

where “const” is the same as in Eq. (3). Then the $\Sigma_{\mu\nu}$ -part of the exponent in Eq. (8) can be represented as an integral over the said auxiliary space-independent electromagnetic field $B_{\mu\nu}$:

$$e^{-G(\lambda)\cdot\Sigma_{\mu\nu}^2} = \frac{1}{[8\pi G(\lambda)]^{3/2}} \left(\prod_{\mu<\nu} \int_{-\infty}^{+\infty} dB_{\mu\nu} \right) e^{-\frac{B_{\mu\nu}^2}{16G(\lambda)} - \frac{i}{2} B_{\mu\nu} \Sigma_{\mu\nu}}.$$

Accordingly, the path integral in Eq. (6) becomes reduced to the so-called Euler–Heisenberg–Schwinger Lagrangian for a scalar particle [11] (for reviews, see [12]):

$$\int_{x_\mu(0)=x_\mu(s)=0} \mathcal{D}x_\mu e^{-\frac{1}{4} \int_0^s d\tau \dot{x}_\mu^2 - \frac{i}{2} B_{\mu\nu} \Sigma_{\mu\nu}} = \frac{1}{(4\pi s)^{3/2}} \frac{abs^2}{\sin(as) \sinh(bs)}. \quad (10)$$

Here a and b can be expressed through the auxiliary electric and magnetic fields corresponding to the Abelian strength tensor $B_{\mu\nu}$ as $a^2 = \frac{1}{2} \left[\mathbf{E}^2 - \mathbf{H}^2 + \sqrt{(\mathbf{E}^2 - \mathbf{H}^2)^2 + 4(\mathbf{E} \cdot \mathbf{H})^2} \right]$ and $b^2 = \frac{1}{2} \left[\mathbf{H}^2 - \mathbf{E}^2 + \sqrt{(\mathbf{E}^2 - \mathbf{H}^2)^2 + 4(\mathbf{E} \cdot \mathbf{H})^2} \right]$. Furthermore, the mass parameter M should be sufficiently large in order for the mean size of the trajectory to be as small as $\mathcal{O}(\nu)$ to ensure the validity of the Gaussian approximation used in the derivation of Eq. (3). Therefore, we set $M \sim \frac{1}{\nu}$, proving afterwards that this parameter is indeed getting exponentially large with temperature. For such large masses in Eq. (6), one can safely expand the Euler–Heisenberg–Schwinger Lagrangian up to the leading nontrivial term as

$$\frac{abs^2}{\sin(as) \sinh(bs)} \simeq 1 + \frac{s^2}{6} \sum_{\mu<\nu} B_{\mu\nu}^2. \quad (11)$$

Then Eqs. (6), (7), (10), and (11) yield for the condensate of the ϕ -field the following intermediate expression:

$$\langle |\phi|^2 \rangle \simeq \frac{1}{\sqrt{3 \cdot 2^{15} \cdot \pi^7}} \int_0^\infty d\lambda \frac{e^{-\lambda^2}}{[G(\lambda)]^{3/2}} \int_0^\infty ds \sqrt{s} e^{-M^2 s} \left(\prod_{\mu<\nu} \int_{-\infty}^{+\infty} dB_{\mu\nu} \right) e^{-\frac{B_{\mu\nu}^2}{16G(\lambda)}} \cdot \sum_{\mu<\nu} B_{\mu\nu}^2.$$

The Gaussian $B_{\mu\nu}$ -integration here can be readily performed as

$$\left(\prod_{\mu<\nu} \int_{-\infty}^{+\infty} dB_{\mu\nu} \right) e^{-\frac{B_{\mu\nu}^2}{16G(\lambda)}} \cdot \sum_{\mu<\nu} B_{\mu\nu}^2 = 4\pi \int_0^\infty dB B^4 e^{-\frac{B^2}{8G(\lambda)}} = 192\sqrt{2}\pi^{3/2} [G(\lambda)]^{5/2},$$

leading to a simple formula

$$\langle |\phi|^2 \rangle \simeq \frac{\sqrt{3}}{4\pi^{3/2} M^3} \int_0^\infty d\lambda G(\lambda) e^{-\lambda^2}.$$

Recalling now the definition of $G(\lambda)$ given in Eq. (9), we obtain

$$\langle |\phi|^2 \rangle \simeq \frac{\text{const}^{3/2}}{12\pi^{3/2}} \frac{\mathcal{E}}{(M\nu)^3} \int_{\lambda_{\min}}^\infty \frac{d\lambda}{\lambda} e^{-\lambda^2}.$$

In order to estimate the cut-off parameter λ_{\min} , we should go back to Eq. (8), where the λ -integration first appeared. The saddle-point value of λ , following from the integral in Eq. (8), reads $\lambda_{\text{s.p.}} \sim \frac{(\mathcal{E}\Sigma_{\mu\nu}^2)^{1/3}}{\nu}$. Since the minimal size of the trajectory is estimated as ν , we conclude that $\lambda_{\text{s.p.}} \gtrsim (\mathcal{E}\nu)^{1/3}$. Therefore, since the λ -integral is “saturated” by the vicinity of $\lambda_{\text{s.p.}}$, we can set $\lambda_{\min} = (\mathcal{E}\nu)^{1/3}$. Recalling that $M \sim \frac{1}{\nu}$, we finally obtain for the condensate of the ϕ -field the following estimate:

$$\langle |\phi|^2 \rangle \propto \mathcal{E} \cdot \ln \frac{1}{\mathcal{E}\nu}. \quad (12)$$

B. A spatially confining dual Landau–Ginzburg theory

Let us now consider a dual Landau–Ginzburg theory, which can describe the phenomenon of spatial confinement. This theory contains a dual Higgs field and a dual gauge field. The dual Higgs field describes a 3-d medium of magnetic monopoles above the temperature of dimensional reduction. Such monopoles are therefore assumed to be those degrees of freedom whose condensation provides spatial confinement of gluons in the dimensionally-reduced high-temperature Yang–Mills plasma.

We consider the simplest, $\text{SU}(2)$, case, and adopt an effective hydrodynamic description of the corresponding dual Landau–Ginzburg theory. Within this description, the dual gauge field is replaced by the velocity field, which is further assumed to be represented as a sum of the low- and the high-energy modes. In a sense, such a separation of the velocity into two components is similar to that of the Landau model of superfluid quantum liquids [13]. In our case, the high-energy modes have the standard, for the Landau–Ginzburg theory, Maxwell-type kinetic term $F_{\mu\nu}^2[v]$, where $F_{\mu\nu}[v] = \partial_\mu v_\nu - \partial_\nu v_\mu$. (Apparently, $F_{\mu\nu}[v]$ is only non-vanishing for a non-potential motion, for which the high-energy part of the velocity cannot be represented as a gradient of some scalar function.) To minimize the number of free parameters, we assume for this kinetic term a normalization factor of $\frac{1}{4\nu}$. Rather, the average over the low-energy modes is given by Eqs. (4) and (5). It yields Kolmogorov turbulence for these modes, and can lead to the formation of the condensate (12).

We make now our main conjecture that the Kolmogorov turbulence of the low-energy velocity modes can be the origin of the dual-Higgs condensation in the dual Landau–Ginzburg theory. As a physical motivation for this conjecture serves the known fact that a conducting turbulent medium expels external magnetic fields [4]. In the presence of a Higgs field, one

can introduce a Meissner-type description of this phenomenon by assuming that turbulence leads to the condensation of this Higgs field. Here we impose a dual version of this scenario, and use Eq. (12) as an estimate for the v.e.v. of the dual Higgs field in the dual Landau–Ginzburg theory:

$$\eta_{3d} \sim \sqrt{\mathcal{E} \cdot \ln \frac{1}{\mathcal{E}\nu}}. \quad (13)$$

We introduce further a phenomenological coupling μ of the dual Higgs field and the high-energy velocity field v_μ . That is, the covariant derivative acting on the dual Higgs field reads $D_\lambda = \partial_\lambda - i\mu v_\lambda$, and the coupling μ has the dimensionality of (mass). Furthermore, we consider the London limit of the dual superconductor. In this limit, the radial part of the dual Higgs field goes over to η_{3d} , and only the phase θ of this field remains dynamical. Accordingly, the action of the dual Landau–Ginzburg theory in the London limit reads

$$S_{3d} = \int d^3x \left\{ \frac{1}{4\nu} F_{\mu\nu}^2[v] + \eta_{3d}^2 (\partial_\mu \theta + \mu v_\mu)^2 \right\}. \quad (14)$$

In order to fix the temperature dependence of μ and ν , we consider now a 4-d dual Abelian Higgs model, which yields the dual Landau–Ginzburg theory upon the high-temperature dimensional reduction. In the London limit of interest, the action of such a dual Abelian Higgs model reads

$$S_{4d} = \int d^4x \left\{ \frac{1}{4} F_{\mu\nu}^2[B] + \eta_{4d}^2 (\partial_\mu \theta + g_m B_\mu)^2 \right\}.$$

Here B_μ is a dual gauge field, $F_{\mu\nu}[B] = \partial_\mu B_\nu - \partial_\nu B_\mu$ is its strength tensor, g_m is a dimensionless magnetic coupling, and we make no difference in notations between the 4-d and the 3-d indices. The action of the corresponding dimensionally-reduced theory follows upon the replacement $\int d^4x \rightarrow \beta \int d^3x$, where $\beta \equiv 1/T$. Matching now the fields and the couplings of the actions S_{4d} and S_{3d} , we obtain the following three relations:

$$\beta \eta_{4d}^2 = \eta_{3d}^2, \quad g_m B_\mu = \mu v_\mu, \quad B_\mu = \sqrt{\frac{T}{\nu}} v_\mu. \quad (15)$$

Let us now consider the soft part of the Yang–Mills gluon plasma which, at sufficiently high temperatures, also undergoes the dimensional reduction [2, 3]. The two quantities characterizing this dimensionally-reduced soft part of the gluon plasma are the spatial string tension σ_s and the inverse spatial vacuum correlation length ξ^{-1} . These quantities can be used to obtain the parametric temperature dependence of μ and ν . To this end, we

notice that, since both σ_s and ξ^{-1} characterize soft stochastic background fields, they are proportional to the corresponding power of g^2T , where g is the finite-temperature Yang–Mills coupling. Namely, one has [3]

$$\sigma_s \propto (g^2T)^2, \quad \xi^{-1} \propto g^2T. \quad (16)$$

On the other hand, in the London limit of a dual superconductor, the string tension is given by the formula $\sigma_s \propto \eta_{4d}^2 \ln \kappa$, where κ is the ratio of the dual-Higgs and the dual-vector-boson masses [14]. We notice that, although the formal derivation of this expression implies the inequality $\ln \kappa \gg 1$, lattice simulations in the Abelian-projected Yang–Mills theory suggest the values of $\ln \kappa \lesssim 5$ (cf. [14] and references therein). Therefore, we have

$$\sigma_s \sim \eta_{4d}^2 = T\eta_{3d}^2 \sim \mathcal{E}T \cdot \ln \frac{1}{\mathcal{E}\nu}, \quad (17)$$

where Eq. (13) has been used at the final step. Using now Eq. (16), we obtain

$$\eta_{3d}^2 \sim \mathcal{E} \cdot \ln \frac{1}{\mathcal{E}\nu} \sim g^4T. \quad (18)$$

(We notice that, for presumably an accidental reason, the right-hand side of this relation coincides with the known ultrasoft scale, the inverse of which is the mean time needed for a parton undergoing Coulomb scatterings in the gluon plasma to deflect by an angle of the order of 1 [15].) Recalling now that $\mathcal{E} = \text{const}$ in the initial Kolmogorov hypothesis, we obtain the sought temperature dependence of the dual kinematic viscosity:

$$\nu = \frac{1}{\mathcal{E}} e^{-\text{const} \cdot \frac{g^4T}{\mathcal{E}}}. \quad (19)$$

Thus, we conclude that the soft contribution to the kinematic viscosity of the dual fluid in our approach is suppressed with temperature as $e^{-\text{const} \cdot \frac{g^4T}{\mathcal{E}}}$. Consequently, the soft contribution to the shear viscosity of the dual fluid is suppressed by the same factor. This finding parallels the smallness of nonperturbative contributions to the dual shear viscosity, which was found within other approaches [5]. We notice that, on the general grounds, perturbative contributions to the shear viscosity are also expected. Those should keep the ratio of the full shear viscosity to the entropy density above the lower $1/(4\pi)$ -bound which, in the isotropic case at issue, is predicted by the AdS/CFT correspondence [16].

Comparing now the inverse vacuum correlation length in the dual Abelian Higgs model with that in the high-temperature Yang–Mills theory, we can get the temperature dependence of the phenomenological coupling μ . Indeed, the inverse vacuum correlation length in

the London limit of the dual superconductor is given by the mass of the dual vector boson, i.e. $\xi^{-1} \sim g_m \eta_{4d}$. The magnetic coupling g_m can be obtained from the second and the third relations (15): $g_m = \mu \sqrt{\beta \nu}$. Using for η_{4d} expression (17), we have $\xi^{-1} \sim \mu \sqrt{\mathcal{E} \nu \cdot \ln \frac{1}{\mathcal{E} \nu}}$. Furthermore, since $\ln \frac{1}{\mathcal{E} \nu} \sim \frac{g^4 T}{\mathcal{E}}$ [cf. Eq. (18)], we can write the previous estimate as $\xi^{-1} \sim g^2 \mu \sqrt{T \nu}$. Equating now this expression to the Yang–Mills one, Eq. (16), we get a simple formula

$$\mu \sim \sqrt{\frac{T}{\nu}}. \quad (20)$$

Using finally Eq. (19), we obtain the temperature dependence of μ in the form

$$\mu \sim \sqrt{\mathcal{E} T} e^{\text{const.} \cdot \frac{g^4 T}{2\mathcal{E}}}, \quad (21)$$

where “const” is the same as in Eq. (19).

Anticipating the results of the next Section, we will encounter there the relation $\mu \nu \neq \frac{1}{\text{integer}}$, which should *not* hold in order for the Aharonov–Bohm-type interactions between the excitations in the dual fluid and the dual Abrikosov vortices to be nontrivial. From Eqs. (19) and (21) we see that

$$\mu \nu \sim \sqrt{\frac{T}{\mathcal{E}}} e^{-\text{const.} \cdot \frac{g^4 T}{2\mathcal{E}}},$$

i.e. it is an exponentially small quantity, which therefore may indeed be $\propto 1/\text{integer}$. This makes the above-quoted constraint on the product $\mu \nu$ nontrivial.

IV. THE AHARONOV–BOHM EFFECT IN THE ABSENCE OF A CHERN–SIMONS TERM

A phase factor (i.e. an unaveraged Wilson loop) associated with an excitation of the dual fluid is defined in the same way as in Eq. (1), i.e. as $\exp(\frac{i}{\nu} \int_x v_\mu j_\mu)$, where from now on we use short-hand notations $\int_x \equiv \int d^3x$ and $\int_p \equiv \int \frac{d^3p}{(2\pi)^3}$. Let us first consider the case without condensation of the dual Higgs field, i.e. set in Eq. (14) $\eta_{3d} = 0$. Then the phase factor averaged with the remaining Maxwell action, $\frac{1}{4\nu} \int_x F_{\mu\nu}^2[v]$, yields the following Wilson loop:

$$\langle W(C) \rangle = \exp \left(-\frac{1}{8\pi\nu} \oint_C dx_\mu \oint_C dy_\mu \frac{1}{|\mathbf{x} - \mathbf{y}|} \right). \quad (22)$$

We consider now the full action (14), with $\eta_{3d} \neq 0$, and average the phase factor $\exp(\frac{i}{\nu} \int_x v_\mu j_\mu)$ with this action. That is, we calculate the following Wilson loop:

$$\langle W(C) \rangle = \int \mathcal{D}v_\mu \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\theta} e^{-\int_x [\frac{1}{4\nu} F_{\mu\nu}^2[v] + \eta^2 (\partial_\mu \theta + \mu v_\mu)^2 - \frac{i}{\nu} v_\mu j_\mu]}. \quad (23)$$

Here, $\eta \equiv \eta_{3d}$, and the full phase θ of the dual Higgs field is represented as a sum $\theta = \tilde{\theta} + \bar{\theta}$, with $\tilde{\theta}$ experiencing jumps by 2π when going around dual Abrikosov vortices, while $\bar{\theta}$ being a Gaussian fluctuation around $\tilde{\theta}$. The said jumps of $\tilde{\theta}$ lead to the noncommutativity of two derivatives acting on this field:

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu) \tilde{\theta} = 2\pi \varepsilon_{\mu\nu\lambda} J_\lambda. \quad (24)$$

Here J_λ is the current of dual Abrikosov vortices, which populate the dual-Higgs vacuum. One can expect that the trajectory C , along which an excitation of the dual fluid evolves, links to the dual Abrikosov vortices. Such a linking should correspond to a long-range Aharonov–Bohm-type interaction between the said excitation and the dual Abrikosov vortices.

To visualize such an Aharonov–Bohm interaction, we perform now a duality transformation of the Wilson loop (23). To this end, it is first convenient to introduce two auxiliary fields as follows:

$$e^{-\frac{1}{4\nu} \int_x F_{\mu\nu}^2} = \int \mathcal{D}G_\mu e^{\int_x [-\frac{\nu}{2} G_\mu^2 + i\varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu G_\lambda]}, \quad e^{-\eta^2 \int_x (\partial_\mu \theta + \mu v_\mu)^2} = \int \mathcal{D}C_\mu e^{\int_x [-\frac{1}{4\eta^2} C_\mu^2 + iC_\mu (\partial_\mu \theta + \mu v_\mu)]}.$$

The subsequent integration over $\bar{\theta}$ leads to the constraint $\partial_\mu C_\mu = 0$, which can be resolved by representing C_μ as $C_\mu = \varepsilon_{\mu\nu\lambda} \partial_\nu \varphi_\lambda$, where the field φ_μ has the dimensionality of (mass). Accordingly, $C_\mu^2 = \frac{1}{2} \Phi_{\mu\nu}^2$, where $\Phi_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu$, and $i \int_x C_\mu \partial_\mu \tilde{\theta} = 2\pi i \int_x \varphi_\mu J_\mu$, where at the last step we have used Eq. (24). Thus, the Wilson loop (23) takes the form

$$\langle W(C) \rangle = \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu \mathcal{D}G_\mu \mathcal{D}v_\mu e^{\int_x [-\frac{\nu}{2} G_\mu^2 - \frac{1}{8\eta^2} \Phi_{\mu\nu}^2 + i\varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu (G_\lambda + \mu \varphi_\lambda) + 2\pi i \varphi_\mu J_\mu + \frac{i}{\nu} v_\mu j_\mu]}. \quad (25)$$

Notice that, throughout this paper, we work at the entirely classical level. For this reason, the Jacobian corresponding to the change of integration variables $\tilde{\theta} \rightarrow J_\mu$ is omitted, and the measure $\mathcal{D}J_\mu$ in the functional integral has only a statistical (rather than a field-theoretical) meaning of counting vortices in their given configuration.

Owing to the representation (25) for the Wilson loop, the v_μ -field becomes a Lagrange multiplier, which leads, upon the integration, to a functional δ -function $\delta(\varepsilon_{\mu\nu\lambda} \partial_\nu (G_\lambda + \mu \varphi_\lambda) + \frac{1}{\nu} j_\mu)$. The G_μ -integration amounts to substituting G_μ , stemming from this δ -function, into $e^{-\frac{\nu}{2} \int_x G_\mu^2}$. Such a G_μ reads $G_\mu = -\mu \varphi_\mu - \frac{1}{\nu} \varepsilon_{\mu\nu\lambda} \int_y \partial_\nu^x D_0^{xy} j_\lambda^y$, where $D_0^{xy} \equiv 1/(4\pi|\mathbf{x} - \mathbf{y}|)$, $j_\lambda^y \equiv j_\lambda(\mathbf{y}; C)$, and the conservation of j_μ has been used. Accordingly, the Wilson loop takes the form

$$\langle W(C) \rangle = \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu e^{\int_x [-\frac{1}{8\eta^2} \Phi_{\mu\nu}^2 + 2\pi i \varphi_\mu J_\mu - \frac{\nu}{2} (\mu \varphi_\mu + \frac{1}{\nu} \varepsilon_{\mu\nu\lambda} \int_y \partial_\nu^x D_0^{xy} j_\lambda^y)^2]},$$

or, equivalently,

$$\langle W(C) \rangle = e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_0^{xy}} \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu e^{\int_x \left(-\frac{1}{8\eta^2} \Phi_{\mu\nu}^2 - \frac{\mu^2 \nu}{2} \varphi_\mu^2 + i\varphi_\mu K_\mu \right)},$$

where

$$K_\mu^x \equiv 2\pi J_\mu^x + i\mu \varepsilon_{\mu\nu\lambda} \int_y \partial_\nu^x D_0^{xy} j_\lambda^y. \quad (26)$$

The remaining φ_μ -integration amounts to introducing a rescaled field $\chi_\mu \equiv \varphi_\mu/(\eta\sqrt{2})$, and denoting

$$m^2 \equiv 2\mu^2 \eta^2 \nu. \quad (27)$$

That yields

$$\int \mathcal{D}\chi_\mu e^{\int_x \left[-\frac{1}{4} (\partial_\mu \chi_\nu - \partial_\nu \chi_\mu)^2 - \frac{m^2}{2} \chi_\mu^2 + i\sqrt{2}\eta \chi_\mu K_\mu \right]} = e^{-\eta^2 \int_{x,y} K_\mu^x K_\mu^y D_m^{xy}},$$

where $D_m^{xy} \equiv e^{-m|\mathbf{x}-\mathbf{y}|}/(4\pi|\mathbf{x}-\mathbf{y}|)$. Thus, the Wilson loop becomes

$$\langle W(C) \rangle = e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_0^{xy}} \int \mathcal{D}J_\mu e^{-\eta^2 \int_{x,y} K_\mu^x K_\mu^y D_m^{xy}}.$$

The last exponent in this formula can be simplified (cf. Appendix A), that yields the following result:

$$\langle W(C) \rangle = e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_m^{xy}} \int \mathcal{D}J_\mu e^{-(2\pi\eta)^2 \int_{x,y} J_\mu^x J_\mu^y D_m^{xy} + \frac{2\pi i}{\mu\nu} [\hat{L}(j,J) - \varepsilon_{\mu\nu\lambda} \int_{x,y} J_\mu^x j_\nu^y \partial_\lambda^x D_m^{xy}]}, \quad (28)$$

where $\hat{L}(j,J) = \varepsilon_{\mu\nu\lambda} \int_{x,y} J_\mu^x j_\nu^y \partial_\lambda^x D_0^{xy}$ is the Gauss' linking number of the contour C and a dual Abrikosov vortex. The corresponding exponent $e^{\frac{2\pi i}{\mu\nu} \hat{L}(j,J)}$ in Eq. (28) describes a long-range Aharonov–Bohm-type interaction of the magnetic excitation with the dual Abrikosov vortex. Thus, this interaction is nontrivial (i.e., $e^{\frac{2\pi i}{\mu\nu} \hat{L}(j,J)} \neq 1$) as long as

$$\mu\nu \neq \frac{1}{\text{integer}}. \quad (29)$$

We notice that, at the formal level, Eq. (28) represents a 3-d counterpart of the corresponding 4-d result for the Wilson loop in the Abelian Higgs model (cf. Ref. [17]).

V. THE AHARONOV–BOHM EFFECT IN THE PRESENCE OF A CHERN–SIMONS TERM

We extend now the analysis of the previous Section to the case where the CS term is included. As a warm-up, we again start with the theory in which the condensation of the

dual Higgs field is absent, i.e. $\eta = 0$. The Wilson loop in such a theory is given by the formula

$$\langle W(C) \rangle = \int \mathcal{D}v_\mu e^{-\int_x [\frac{1}{4\nu} F_{\mu\nu}^2[v] + i\Theta \varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu v_\lambda - \frac{i}{\nu} v_\mu j_\mu]},$$

where the dimensionality of the new parameter Θ is (mass)². Imposing the gauge-fixing condition $\partial_\mu v_\mu = 0$, we obtain the saddle-point equation

$$-\partial^2 v_\mu + im\varepsilon_{\mu\nu\lambda} \partial_\nu v_\lambda = ij_\mu, \quad \text{where} \quad m = 2\Theta\nu.$$

Seeking a solution to this saddle-point equation in the form $v_\mu = U_\mu + iV_\mu$, we get a system of equations

$$\partial^2 U_\mu + m\varepsilon_{\mu\nu\lambda} \partial_\nu V_\lambda = 0, \quad -\partial^2 V_\mu + m\varepsilon_{\mu\nu\lambda} \partial_\nu U_\lambda = j_\mu. \quad (30)$$

The first of these equations can be solved with respect to U_μ as

$$U_\mu^x = m\varepsilon_{\mu\nu\lambda} \int_y D_0^{xy} \partial_\nu^y V_\lambda^y. \quad (31)$$

Differentiating the second equation (30), and applying the maximum principle, one gets $\partial_\mu V_\mu = 0$. Using this relation, one further obtains from Eq. (31): $\varepsilon_{\mu\nu\lambda} \partial_\nu U_\lambda = mV_\mu$. Substitution of this formula into the second equation (30) yields for that equation a remarkably simple form $(-\partial^2 + m^2)V_\mu = j_\mu$. Therefore, one has $V_\mu^x = \int_y D_m^{xy} j_\mu^y$, while U_μ^x , given by Eq. (31), can be calculated by virtue of Eq. (A2), and reads $U_\mu^x = \frac{1}{m} \varepsilon_{\mu\nu\lambda} \int_y (D_0^{xy} - D_m^{xy}) \partial_\nu^y j_\lambda^y$. Altogether, the resulting Wilson loop has the form

$$\langle W(C) \rangle|_{\eta=0} = \exp \left\{ \frac{1}{2\nu} \int_{x,y} \left[-j_\mu^x D_m^{xy} j_\mu^y + \frac{i}{m} \varepsilon_{\mu\nu\lambda} j_\mu^x j_\lambda^y \partial_\nu^x (D_0^{xy} - D_m^{xy}) \right] \right\}. \quad (32)$$

This expression generalizes Eq. (22) to the case of $\Theta \neq 0$. Clearly, the Θ -term leads to the appearance of a self-linking of the contour C , as well as of a short-range self-interaction of this contour by means of the Yukawa propagator D_m^{xy} . We also notice that, when $\Theta \rightarrow 0$ in Eq. (32), Eq. (22) is recovered. Indeed, in this limit, one has $\frac{1}{m}(D_0^{xy} - D_m^{xy}) \rightarrow \frac{1}{4\pi}$, so that

$$\frac{1}{m} \int_{x,y} j_\mu^x j_\lambda^y \partial_\nu^x (D_0^{xy} - D_m^{xy}) = \frac{1}{m} \int_{x,y} j_\mu^x (D_0^{xy} - D_m^{xy}) \partial_\nu^y j_\lambda^y \rightarrow \frac{1}{4\pi} \int_{x,y} j_\mu^x \partial_\nu^y j_\lambda^y = 0,$$

since $\int_x j_\mu^x = 0$.

We proceed now to the duality transformation of the Wilson loop in a full theory, where the condensation of the dual Higgs field does take place, i.e. $\eta \neq 0$. The corresponding generalization of Eq. (23) reads

$$\langle W(C) \rangle = \int \mathcal{D}v_\mu \mathcal{D}\tilde{\theta} \mathcal{D}\bar{\theta} e^{-\int_x [\frac{1}{4\nu} F_{\mu\nu}^2[v] + \eta^2 (\partial_\mu \theta + \mu v_\mu)^2 + i\Theta \varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu v_\lambda - \frac{i}{\nu} v_\mu j_\mu]}, \quad (33)$$

The transformation leading from Eq. (23) to Eq. (25) remains the same, so that the counterpart of Eq. (25) in the presence of the CS term reads

$$\langle W(C) \rangle = \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu \mathcal{D}G_\mu \mathcal{D}v_\mu e^{\int_x \left\{ -\frac{\nu}{2}G_\mu^2 - \frac{1}{8\eta^2}\Phi_{\mu\nu}^2 + iv_\mu [\varepsilon_{\mu\nu\lambda}\partial_\nu(G_\lambda + \mu\varphi_\lambda - \Theta v_\lambda) + \frac{1}{\nu}j_\mu] + 2\pi i\varphi_\mu J_\mu \right\}}.$$

Because of the CS term, v_μ now ceases to be a Lagrange multiplier. Nevertheless, since the v_μ -integration is Gaussian, it can be performed exactly.

The corresponding saddle-point equation for v_μ reads $\varepsilon_{\mu\nu\lambda}\partial_\nu v_\lambda = \frac{1}{2\Theta}k_\mu$, where we have denoted $k_\mu = \varepsilon_{\mu\nu\lambda}\partial_\nu(G_\lambda + \mu\varphi_\lambda) + \frac{1}{\nu}j_\mu$. Owing to the conservation of k_μ , a solution to this saddle-point equation reads $v_\mu^x = \frac{1}{2\Theta}\varepsilon_{\mu\nu\lambda}\partial_\nu^x \int_y D_0^{xy} k_\lambda^y$. Plugging this solution back into the exponent $e^{i\int_x v_\mu(k_\mu - \Theta\varepsilon_{\mu\nu\lambda}\partial_\nu v_\lambda)}$, and using the above explicit expression for k_μ , we obtain, upon some algebra, the following formula for the Wilson loop:

$$\begin{aligned} \langle W(C) \rangle &= e^{\frac{i}{2\nu m}\varepsilon_{\mu\nu\lambda}\int_{x,y} j_\mu^x j_\lambda^y \partial_\nu^x D_0^{xy}} \times \\ &\times \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu \mathcal{D}G_\mu e^{\int_x \left\{ -\frac{\nu}{2}G_\mu^2 - \frac{1}{8\eta^2}\Phi_{\mu\nu}^2 + \frac{i}{4\Theta}\varepsilon_{\mu\nu\lambda}[G_\mu\partial_\nu(G_\lambda + 2\mu\varphi_\lambda) + \mu^2\varphi_\mu\partial_\nu\varphi_\lambda] + \frac{i}{2\Theta\nu}(G_\mu + \mu\varphi_\mu)j_\mu + 2\pi i\varphi_\mu J_\mu \right\}}. \end{aligned} \quad (34)$$

Here, the argument of the first exponent coincides with the term containing the Gauss' self-linking number of the contour C , which was present already in Eq. (32). In general, such self-linkings are always induced by the CS term [18]. In addition, the functional integral in Eq. (34) describes interactions of a dual-fluid excitation, which evolves along the contour C , with the dual Abrikosov vortices, as well as self-interactions of this excitation in the presence of the CS term.

In order to visualize all these interactions, let us perform the G_μ -integration first. Representing the saddle-point expression for G_μ in the form $G_\mu = L_\mu + iN_\mu$, we obtain a system of two saddle-point equations:

$$\varepsilon_{\mu\nu\lambda}\partial_\nu L_\lambda - mN_\mu + n_\mu = 0, \quad \varepsilon_{\mu\nu\lambda}\partial_\nu N_\lambda + mL_\mu = 0,$$

where we have denoted $n_\mu = \mu\varepsilon_{\mu\nu\lambda}\partial_\nu\varphi_\lambda + \frac{1}{\nu}j_\mu$. Owing to the conservation of n_μ , we find a solution to these equations in the form

$$L_\mu^x = -\varepsilon_{\mu\nu\lambda}\int_y D_m^{xy}\partial_\nu^y n_\lambda^y, \quad N_\mu^x = m\int_y D_m^{xy}n_\mu^y.$$

Plugging the corresponding saddle-point expression for G_μ back into Eq. (34), we obtain, after some algebra, the following general result:

$$\int \mathcal{D}G_\mu e^{\int_x \left(-\frac{\nu}{2}G_\mu^2 + \frac{i}{4\Theta}\varepsilon_{\mu\nu\lambda}G_\mu\partial_\nu G_\lambda + \frac{i}{2\Theta}G_\mu k_\mu \right)} =$$

$$\begin{aligned}
&= e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_m^{xy} - \mu \varepsilon_{\mu\nu\lambda} \int_{x,y} D_m^{xy} j_\mu^x \partial_\nu^y \varphi_\lambda^y + \frac{\nu \mu^2}{2} \left[\int_{x,y} D_m^{xy} \cdot (m^2 \varphi_\mu^x \varphi_\mu^y + \partial_\mu^x \varphi_\mu^x \cdot \partial_\nu^y \varphi_\nu^y) - \int_x \varphi_\mu^2 \right]} \times \\
&\times e^{-\frac{i}{4\Theta} \left\{ \mu^2 \varepsilon_{\mu\nu\lambda} \int_x \varphi_\mu \partial_\nu \varphi_\lambda + \varepsilon_{\mu\nu\lambda} \int_{x,y} D_m^{xy} \cdot \left[\frac{1}{\nu^2} j_\mu^x \partial_\nu^y j_\lambda^y - (\mu m)^2 \varphi_\mu^x \partial_\nu^y \varphi_\lambda^y \right] + \frac{2\mu}{\nu} \left(\int_x \varphi_\mu j_\mu - m^2 \int_{x,y} D_m^{xy} \varphi_\mu^x j_\mu^y \right) \right\}}. \quad (35)
\end{aligned}$$

For exponentially small ν 's of interest [cf. Eq. (19)], the remaining φ_μ -integration in Eq. (34) should reproduce Eq. (22), which follows directly from the initial Eq. (33):

$$\langle W(C) \rangle \rightarrow \int \mathcal{D}v_\mu e^{-\int_x \left(\frac{1}{4\nu} F_{\mu\nu}^2 - \frac{i}{\nu} v_\mu j_\mu \right)} = e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_0^{xy}}. \quad (36)$$

Therefore, the $(\nu \rightarrow 0)$ -limit can serve as a check for the above-performed calculation. The right-hand side of Eq. (35) simplifies in this limit to the form

$$e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_0^{xy} - \mu \varepsilon_{\mu\nu\lambda} \int_{x,y} D_0^{xy} j_\mu^x \partial_\nu^y \varphi_\lambda^y - \frac{i}{4\Theta} \left(\mu^2 \varepsilon_{\mu\nu\lambda} \int_x \varphi_\mu \partial_\nu \varphi_\lambda + \frac{1}{\nu^2} \varepsilon_{\mu\nu\lambda} \int_{x,y} D_0^{xy} j_\mu^x \partial_\nu^y j_\lambda^y + \frac{2\mu}{\nu} \int_x \varphi_\mu j_\mu \right)},$$

and the Wilson loop (34) becomes

$$\begin{aligned}
\langle W(C) \rangle &\rightarrow e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_0^{xy} + \frac{i}{4\Theta \nu^2} \varepsilon_{\mu\nu\lambda} \int_{x,y} (j_\mu^x j_\lambda^y \partial_\nu^x D_0^{xy} - D_0^{xy} j_\mu^x \partial_\nu^y j_\lambda^y)} \times \\
&\times \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu e^{\int_x \left(-\frac{1}{8\eta^2} \Phi_{\mu\nu}^2 + 2\pi i \varphi_\mu J_\mu \right) - \mu \varepsilon_{\mu\nu\lambda} \int_{x,y} \varphi_\mu^x j_\lambda^y \partial_\nu^x D_0^{xy}}.
\end{aligned}$$

The Gaussian φ_μ -integration in this formula yields, upon some algebra,

$$\langle W(C) \rangle \rightarrow e^{-\frac{1}{2\nu} \int_{x,y} j_\mu^x j_\mu^y D_0^{xy}} \int \mathcal{D}J_\mu e^{-(2\pi\eta)^2 \int_{x,y} J_\mu^x J_\mu^y D_0^{xy}}.$$

Recalling the normalization of the integration measure $\mathcal{D}J_\mu$, discussed in Appendix A, we indeed recover the expected result (36). Thus, our check of Eq. (35) was successful.

We consider now large values of the Θ -parameter, namely such that

$$\Theta \gg \frac{g^2 T}{\nu} = g^2 T \mathcal{E} \cdot e^{\text{const} \cdot \frac{g^4 T}{\mathcal{E}}}, \quad (37)$$

where Eq. (19) has been used at the final step. For such Θ 's, the action in the exponents on the right-hand side of Eq. (35) becomes local, and the Wilson loop (34) gets simplified to the form

$$\begin{aligned}
\langle W(C) \rangle &\rightarrow e^{-\frac{1}{2\nu m^2} \int_x j_\mu^2 + \frac{i}{4\Theta \nu^2} \varepsilon_{\mu\nu\lambda} \left(\int_{x,y} j_\mu^x j_\lambda^y \partial_\nu^x D_0^{xy} - \frac{1}{m^2} \int_x j_\mu \partial_\nu j_\lambda \right)} \times \\
&\times \int \mathcal{D}J_\mu \mathcal{D}\varphi_\mu e^{\int_x \left[-\frac{1}{8\eta^2} \Phi_{\mu\nu}^2 + \frac{\mu^2}{8\Theta^2 \nu} (\partial_\mu \varphi_\mu)^2 + \frac{i\mu^2}{4\Theta} \varepsilon_{\mu\nu\lambda} \varphi_\mu \partial_\nu \varphi_\lambda + i\varphi_\mu \left(2\pi J_\mu + \frac{\mu}{m} j_\mu + \frac{i\mu}{m^2} \varepsilon_{\mu\nu\lambda} \partial_\nu j_\lambda \right) \right]}. \quad (38)
\end{aligned}$$

Furthermore, the φ_μ -integration in this formula can also be performed analytically in the same limiting case (37). Referring the reader for the details to Appendix B, we present here the final result of this integration:

$$\langle W(C) \rangle \rightarrow e^{-\frac{1}{2\nu m^2} \int_x j_\mu^2 - \frac{i\Theta}{m^4} \varepsilon_{\mu\nu\lambda} \int_x j_\mu \partial_\nu j_\lambda} \times$$

$$\times \int \mathcal{D}J_\mu e^{-\eta^2 \int_{x,y} R_\mu^x R_\mu^y D_{\mathcal{M}}^{xy} + \frac{i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} \int_{x,y} [R_\mu^x R_\lambda^y \partial_\nu D_{\mathcal{M}}^{xy} - 4\pi J_\mu^x (\pi J_\lambda^y + \frac{\mu}{m} j_\lambda^y) \partial_\nu D_0^{xy}]}. \quad (39)$$

In this formula, $\mathcal{M} \equiv \frac{\mu^2 \eta^2}{\Theta}$, and $R_\mu \equiv 2\pi J_\mu + \frac{\mu}{m} j_\mu$. Remarkably, in the limit (37), the initial CS term for the velocity, $i\Theta \varepsilon_{\mu\nu\lambda} v_\mu \partial_\nu v_\lambda$, leads to the appearance of its counterpart $\frac{i\Theta}{m^4} \varepsilon_{\mu\nu\lambda} j_\mu \partial_\nu j_\lambda$ for the current j_μ , while the self-linking of the contour C , described by the first exponent in Eq. (34), disappears. Rather, we observe the appearance of self-linkings of the dual Abrikosov vortices, as well as their linkings with the contour C , as described by the term $\frac{4\pi i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} J_\mu^x (\pi J_\lambda^y + \frac{\mu}{m} j_\lambda^y) \partial_\nu D_0^{xy}$ in the Lagrangian. In particular, its part $\frac{4\pi i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} J_\mu^x \cdot \frac{\mu}{m} j_\lambda^y \partial_\nu D_0^{xy}$ yields the term $-\frac{2\pi i}{\mu\nu} \hat{L}(j, J)$ in the action. Thus, we conclude that, in the presence of the CS term, the Aharonov–Bohm-type interaction of the dual excitation with the dual Abrikosov vortex is nontrivial in the limit (37) as long as

$$\mu\nu \neq \frac{1}{\text{integer}}. \quad (40)$$

Remarkably, this condition coincides with Eq. (29), which was obtained in the absence of the CS term. In a similar way, the above term $\frac{4\pi^2 i\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} J_\mu^x J_\lambda^y \partial_\nu D_0^{xy}$ means that the CS term makes dual Abrikosov vortices knotted unless $\frac{\mu^2}{\Theta} \neq \frac{2\pi}{\text{integer}}$.

VI. SUMMARY

The 't Hooft–Mandelstam scenario models spatial confinement in the SU(2) gluon plasma by means of a dual Landau–Ginzburg theory. We have used an effective hydrodynamic description, where the dual-Higgs condensate is formed owing to a turbulent fluid, which respects the Kolmogorov scaling law. Such an approach yields an estimate for the v.e.v. of the dual Higgs field, Eq. (13). Using this finding along with the known expressions for the spatial string tension and the correlation length in the high-temperature Yang–Mills theory, we have found the parametric temperature-dependence of the dual kinematic viscosity ν and the phenomenological coupling μ of the dual Higgs and velocity fields, Eqs. (19) and (21). In particular, the product of these two quantities comes out exponentially suppressed with temperature, and hence parametrically smaller than unity.

We have further considered a Wilson loop describing an excitation in the dual fluid. Such an excitation is involved, in particular, in a long-range Aharonov–Bohm-type interaction with the dual Abrikosov vortices, which are present in the dual Landau–Ginzburg theory. We have shown that this interaction is nontrivial for all but discrete values (29) of the

product $\mu\nu$. These values are smaller than unity, much as the calculated product $\mu\nu$ in general (cf. the previous paragraph), that makes constraint (29) nontrivial.

The Wilson loop describing an excitation of the dual fluid has further been calculated in the presence of a CS term. In the absence of the dual Higgs field, the result is given by Eq. (32), which includes self-linking of the contour of the Wilson loop. In the presence of the dual Higgs field, the result is given by Eqs. (34) and (35), where the remaining φ_μ -integration can be performed analytically only in some limiting cases. In the case of a vanishingly small viscosity ν at fixed Θ , the expected result (36) is reproduced. In another limiting case, when Θ 's are as large as those given by Eq. (37), the φ_μ -integration can also be performed analytically, yielding Eq. (39). This result contains terms, which describe self-linking of dual Abrikosov vortices, as well as their linking with the contour of the Wilson loop. The latter linking represents the same Aharonov–Bohm effect as in the absence of a CS term. In particular, the critical discrete values of the product $\mu\nu$, for which this effect disappears, are given by Eq. (40), that coincides with its counterpart (29) obtained without the CS term. Rather, the appearance of knotted dual Abrikosov vortices is a qualitatively novel effect, which does not exist in the absence of a CS term.

Notice finally that the topological results obtained in this paper can be generalized to the cases of SU(3)- and SU(N_c)-inspired dual Landau–Ginzburg-type theories along the lines of Ref. [19].

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Appendix A: Some details of the derivation of Eq. (28)

With the use of Eq. (26), and owing to the conservation of j_μ , one has

$$-\eta^2 \int_{x,y} K_\mu^x K_\mu^y D_m^{xy} = -(2\pi\eta)^2 \int_{x,y} J_\mu^x J_\mu^y D_m^{xy} - 4\pi i \mu \eta^2 \varepsilon_{\mu\nu\lambda} \int_{x,y} D_m^{xy} J_\mu^x \partial_\nu^y \int_z D_0^{yz} j_\lambda^z +$$

$$+ (\mu\eta)^2 \int_{x,y} D_{\mathbf{m}}^{xy} \left(\partial_{\nu}^x \int_z D_0^{xz} j_{\lambda}^z \right) \left(\partial_{\nu}^y \int_u D_0^{yu} j_{\lambda}^u \right). \quad (\text{A1})$$

We furthermore assume the standard normalization $\langle 1 \rangle = 1$ of the functional average, which implies a division by the functional integral $\int \mathcal{D}J_{\mu} e^{-(2\pi\eta)^2 \int_{x,y} J_{\mu}^x J_{\mu}^y D_{\mathbf{m}}^{xy}}$ corresponding to the first term on the right-hand side of Eq. (A1). Thus, we always imply that the measure $\mathcal{D}J_{\mu}$ is normalized by a division by this integral.

The last term in Eq. (A1) can be represented, through the integration by parts, as $(\mu\eta)^2 \int_{x,y} D_{\mathbf{m}}^{xy} j_{\mu}^x \int_u D_0^{yu} j_{\mu}^u$. The y -integration in this expression is straightforward:

$$\int_y D_{\mathbf{m}}^{xy} D_0^{yu} = \int_y \int_p \frac{e^{i\mathbf{p}(\mathbf{x}-\mathbf{y})}}{\mathbf{p}^2 + \mathbf{m}^2} \int_q \frac{e^{i\mathbf{q}(\mathbf{y}-\mathbf{u})}}{\mathbf{q}^2} = \int_p \frac{e^{i\mathbf{p}(\mathbf{x}-\mathbf{u})}}{\mathbf{p}^2(\mathbf{p}^2 + \mathbf{m}^2)} = \frac{1}{\mathbf{m}^2} (D_0^{xu} - D_{\mathbf{m}}^{xu}), \quad (\text{A2})$$

where the equality $\frac{1}{\mathbf{p}^2(\mathbf{p}^2 + \mathbf{m}^2)} = \frac{1}{\mathbf{m}^2} \left(\frac{1}{\mathbf{p}^2} - \frac{1}{\mathbf{p}^2 + \mathbf{m}^2} \right)$ has been used at the last step. Using further the explicit form of \mathbf{m}^2 , Eq. (27), we can represent the last term in Eq. (A1) as $\frac{1}{2\nu} \int_{x,y} j_{\mu}^x j_{\mu}^y (D_0^{xy} - D_{\mathbf{m}}^{xy})$.

In the second term on the right-hand side of Eq. (A1), one can use the equality $\partial_{\nu}^y \int_z D_0^{yz} j_{\lambda}^z = \int_z D_0^{yz} \partial_{\nu}^z j_{\lambda}^z$, which yields the same y -integration as in Eq. (A2): $\int_y D_{\mathbf{m}}^{xy} D_0^{yz} = \frac{1}{\mathbf{m}^2} (D_0^{xz} - D_{\mathbf{m}}^{xz})$. Upon the subsequent integration by parts, we obtain for this term the following expression: $\frac{2\pi i}{\mu\nu} \varepsilon_{\mu\nu\lambda} \int_{x,y} J_{\mu}^x j_{\nu}^y \partial_{\lambda}^x (D_0^{xy} - D_{\mathbf{m}}^{xy})$. Noticing also the definition of the Gauss' linking number, $\hat{L}(j, J) = \varepsilon_{\mu\nu\lambda} \int_{x,y} J_{\mu}^x j_{\nu}^y \partial_{\lambda}^x D_0^{xy}$, we arrive at Eq. (28).

Appendix B: Some details of the derivation of Eq. (39)

For Θ 's obeying condition (37), one can use relations (18) and (20) to obtain the inequality

$$\frac{\mu^2}{\Theta^2 \nu} \ll \frac{1}{\eta^2}. \quad (\text{B1})$$

Owing to this inequality, the term $\frac{\mu^2}{8\Theta^2 \nu} (\partial_{\mu} \varphi_{\mu})^2$ in Eq. (38) can be neglected in comparison with the absolute value of the term $-\frac{1}{8\eta^2} \Phi_{\mu\nu}^2$. The resulting Gaussian φ_{μ} -integration can be performed by seeking the saddle-point function in the form $\varphi_{\mu} = \varphi_{\mu}^{(1)} + i\varphi_{\mu}^{(2)}$, and solving the so-emerging system of equations for $\varphi_{\mu}^{(1)}$ and $\varphi_{\mu}^{(2)}$. The result can be written as

$$\int \mathcal{D}\varphi_{\mu} e^{\int_x [\dots]} = e^{\frac{1}{2} \int_x [-R_{\mu} \varphi_{\mu}^{(2)} - S_{\mu} \varphi_{\mu}^{(1)} + i(R_{\mu} \varphi_{\mu}^{(1)} - S_{\mu} \varphi_{\mu}^{(2)})]}, \quad (\text{B2})$$

where $R_{\mu} \equiv 2\pi J_{\mu} + \frac{\mu}{m} j_{\mu}$, $S_{\mu} \equiv \frac{\mu}{m^2} \varepsilon_{\mu\nu\lambda} \partial_{\nu} j_{\lambda}$ are respectively the real and the imaginary parts of the current which couples to φ_{μ} in Eq. (38). The obtained real and imaginary parts of

the saddle-point function φ_μ entering Eq. (B2) read

$$\begin{aligned} \varphi_\mu^{(1)} = \\ = 2\mu\eta^2\varepsilon_{\mu\nu\lambda} \left\{ \frac{\mu\eta^2}{\Theta\mathcal{M}^2} \int_y \left[2\pi J_\lambda^y + \frac{\mu}{m} \left(1 - \frac{\mathcal{M}}{m} \right) j_\lambda^y \right] \partial_\nu^x (D_\mathcal{M}^{xy} - D_0^{xy}) - \frac{1}{m^2} \int_y j_\lambda^y \partial_\nu^x D_0^{xy} \right\} \end{aligned} \quad (\text{B3})$$

and

$$\varphi_\mu^{(2)} = 2\eta^2 \int_y D_\mathcal{M}^{xy} \left[2\pi J_\mu^y + \frac{\mu}{m} \left(1 - \frac{\mathcal{M}}{m} \right) j_\mu^y \right], \quad (\text{B4})$$

with the new mass parameter $\mathcal{M} \equiv \frac{\mu^2\eta^2}{\Theta}$. Furthermore, in the limit (B1) at issue, the $\mathcal{O}(\mathcal{M}/m)$ -terms in Eqs. (B3) and (B4) should be neglected compared to 1. That yields the following saddle-point expressions for $\varphi_\mu^{(1)}$ and $\varphi_\mu^{(2)}$:

$$\varphi_\mu^{(1)} = \frac{2\Theta}{\mu^2} \varepsilon_{\mu\nu\lambda} \int_y R_\lambda^y \partial_\nu^x (D_\mathcal{M}^{xy} - D_0^{xy}), \quad \varphi_\mu^{(2)} = 2\eta^2 \int_y R_\mu^y D_\mathcal{M}^{xy}.$$

Substituting them into Eq. (B2), we obtain for the Wilson loop in the limit (37) expression (39).

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